SCIENTIFIC ANALYSIS OF QUALITATIVE CHANGE

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1. Introduction

MODERN SCIENCE has come to the firm conclusion that the material world is composed of matter, and all matter is constantly undergoing *change* in the state of existence. This fact is by no means self evident. Indeed, people in the olden times believed that there are things that change, and things that do not. The moon, the sun, stars in the sky, and mighty mountains on earth do not seem to change at all. True, the moon changes through phases — but that also is precisely repetitive, and we do not see any change in the cyclic repetition of the phases. The biological entities — the deer, the lion and the monkey — do change within one animal's lifetime, but the character of the species itself did not seem to change. One's grandfather's grandfather took the same kind of cows to graze, the character of the cow did not seem to change.

Slowly, with the progress of scientific investigation following Renaissance, evidences accumulated indicating that what we believed as unchangeable also do change. Fossil records indicated that species change, geological investigations revealed that mountains form and evolve, astronomical investigations revealed that changes also do occur in stars, nebulas and galaxies.

Philosophers then set about the task of integrating the everchanging nature of the material world into a system of scientific philosophy. The first attempt came from the German philosopher Hegel, who showed that though most of the time material entities undergo slow and quantitative changes, there are times when the changes become drastic and qualitative — transforming one state of existence into another. The transformation of water into ice, of a seed into a plant, etc., are commonplace examples of such qualitative change of one state of existence into another.

When Marx and Engels tried to create a scientific philosophy by generalizing upon and integrating the knowledge earned through particular branches of science, the general observation of the everchanging character of matter and its progress from quantitative change to qualitative change, and vice versa, became integrated into the newly emerging scientific philosophy — dialectical materialism.

Since then, various branches of science have investigated the process of change of particular material entities or systems. Since matter exists in motion, the changes can be of two broad categories: (1) changes in the state of the matter, and (2) changes in the state of motion. Both types are characterized by quantitative as well as qualitative changes. Studies on the qualitative changes in the *state of matter* have resulted into a field of knowledge which in scientific perlance is known as "phase transition and critical phenomena." And qualitative changes in the *state of motion* is studied in a field known as "bifurcation theory."

Till about thirty years back, one could

explain qualitative changes in motion only in terms of philosophical arguments. But since then a sound mathematical theory has been developed and enough empirical observations have accumulated to back up the philosophical proposition with hard science. In the present article we shall discuss what is now known to be the process of qualitative change in the *state of motion*. For that, we would have to delve into a bit of mathematics.

2. States of motion

Scientists have devised a very intuitively appealing and pictorial way of representing motion. Here, the word "motion" is used in a general sense, meaning any kind of change in the dynamical state of the system. The change of magnetic field in an electromagnet (which itself may not move) is also treated as a change of dynamical state. Likewise, the voltage of an electrical circuit also represents a dynamical state, though the components of the circuit itself may be static.

In general, the dynamical state of a system is represented by a few *variables*. These are a few quantities that suffice to define the dynamical status of a system uniquely. For example, for a pendulum moving in a plane, the anglular position of the bob (θ) and its angular velocity ($\dot{\theta}$) are the two quantities that uniquely represent the dynamical status of the pendulum. When you throw a ball in air, the position of the ball (given by the x, y, and z coordinates) and its velocity (given by the time derivatives of these coordinates) uniquely define the dynamical state of a system. As the ball moves, these quantities vary and so these are called "variables." In any dynamical system, the minimum number of variables that uniquely define the state of the system are called the "state variables."

To pictorially represent the dynamics of a system, scientists imagine a space with the state variables as coordinates. This is called the "state space" or "phase space." Thus, to represent the motion of the pendulum's bob, one would draw a graph with θ and $\dot{\theta}$ as the coordinates. The state of the system at any moment of time is represented by a point in this state space. *Motion* is then represented by the change of position of this state-point. Thus, when a system undergoes dynamical change — quantitative or qualitative — it is then represented by the *trajectory* or *orbit* of the state point in the state space.

How does one obtain this trajectory? Answering this question, exactly, was the main contribution of Isaac Newton. He showed that we can calculate the trajectory if we know how the states change. That is, if we obtain a set of mathematical relationships in the form

$$\begin{aligned} \frac{dx}{dt} = \dot{x} &= \text{a function of } x \text{ and } y \\ \frac{dy}{dt} = \dot{y} &= \text{another function of } x \text{ and } y \end{aligned}$$

then we can calculate the trajectory starting from any initial condition.

Newton's laws enable one to obtain such equations for mechanical systems, from which the future evolution of any system can be calculated. Scientists in other fields formulated similar laws to obtain such "differential equations" — Kirchoff's laws for electrical circuits, Maxwell's equations for electromagnetic fields, Schrödinger's equation in quantum mechanics, etc.

Given a dynamical system, we can thus obtain differential equations, and solving the equations we obtain the trajectory in the state space. Having achieved this ability, man turned to understanding mathematically the quantitative and qualitative changes in motion in terms of the quantitative and qualitative changes in the *character of the trajectories*.

3. Different types of trajectories

Since our objective here is to understand the *qualitative* changes, we need to understand what is meant by fundamentally different types of dynamical behaviour, i.e., different types of trajectories in the state space. For this, let us consider the trajectory of the simple pendulum without air friction. If we move the bob to some angle and release it, the initial condition will be θ = some value and $\dot{\theta} = 0$. The subsequent oscillation of the pendulum will make θ and $\dot{\theta}$ vary periodically between a positive value and a negative value. Moreover, when θ is zero, $\dot{\theta}$ is non-zero (bottom position) and vice versa. This implies that the trajectory in the state-space will be a closed orbit as shown in Fig. 1.

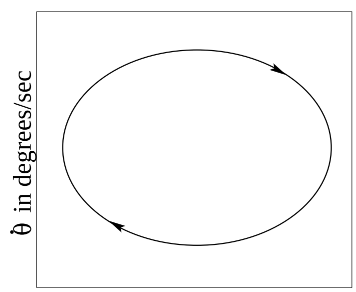


Figure 1: The state-space trajectory of a simple pendulum.

If we consider air friction, the oscillation will slowly die down and the bob will finally settle in the vertically-downward position. In the state space this "damped oscillation" is represented by an inward spiralling motion (Fig. 2). If the friction is too high (for example, if the pendulum is moving in a viscous fluid), there will be no oscillation and the bob will move straight towards the vertical position. In the state space we will see a point moving to an equilibrium position without any spiralling motion.

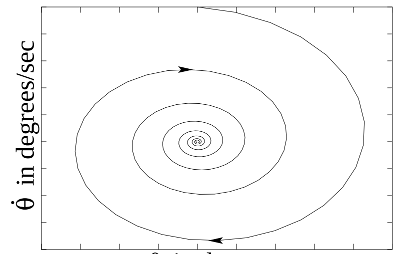
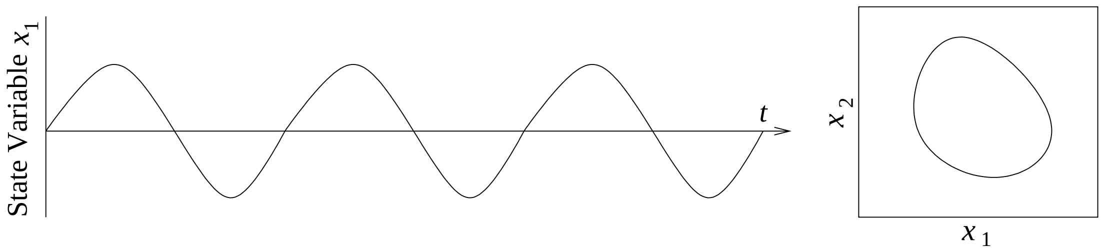


Figure 2: The state-space trajectory of a simple pendulum with friction.

Thus we see examples of three simple but *different* kinds of motion in the state space. Simple, because the evolutions in these cases are guided by linear differential equations. Where the differential equations are nonlinear (a vast majority of dynamical systems found in nature) more complicated situations may arise.

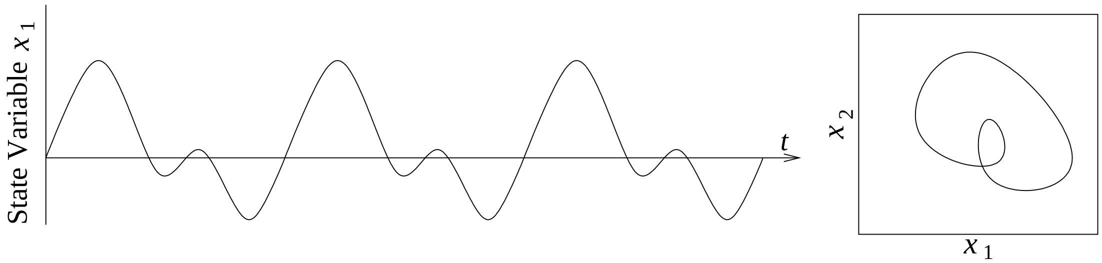
For example, consider any regular rhythmic motion found in nature. Our heart rhythm is a good example, and a little reflection will convince you that there are many such "oscillators" in nature, including the human body. Their trajectory in the state space is also a closed loop, but with one major difference with that of the frictionless simple pendulum. In the case of a pendulum, if you move the bob to a larger angle before releasing it (a different initial condition), the state-point moves in a different closed loop with a larger diameter. This means, if you perturb the state the system settles into a different trajectory. But the human heart cannot afford to do that. You may be startled by the sudden burst of a cracker — setting the heart into an accelerated pace — but after some time the rhythm comes back to its original one, the original closed loop in the state space. Any perturbation from this loop eventually dies down

Figure 3: A few fundamentally different types of orbits.



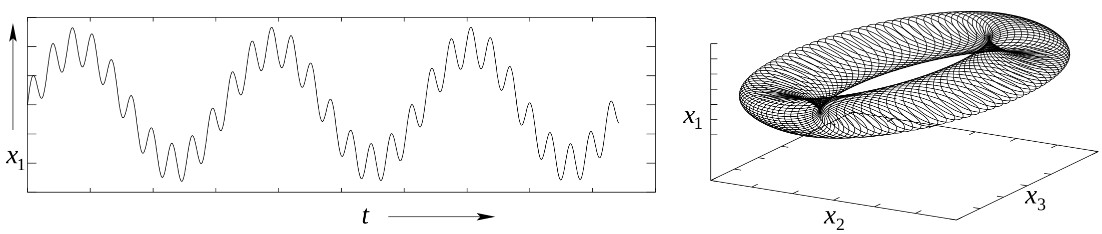
(a)

The time plot (left) and the state space trajectory (right) for a period-1 attractor.



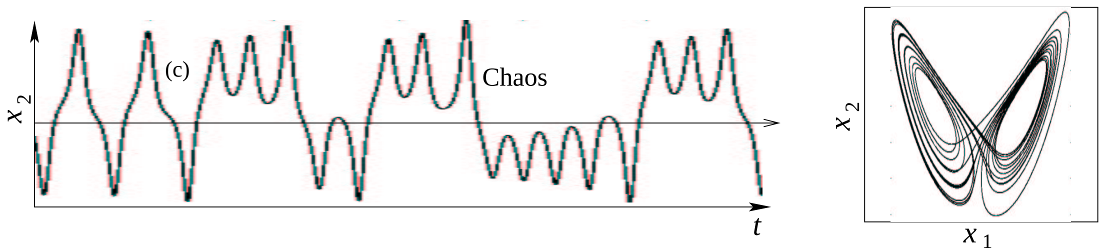
(b)

The time plot (left) and the state space trajectory (right) for a period-2 attractor.



(c)

The time plot (left) and the state space trajectory (right) for a quasiperiodic attractor.



(d)

The time plot (left) and the state space trajectory (right) for a chaotic attractor.

and the state is *attracted* to a particular orbit. Such stable closed-loop orbits, therefore, are called "attractors." Quite diverse things — starting from the oscillator that sets the "clock" inside a computer to the return of Olive-Ridley turtles to the Orissa beach at a specific time of the year — are examples such periodic attractors.

In some cases the periodicity may be more than one. If a system comes back to the same state after two oscillation-cycles, it is called a period-2 trajectory (See Fig. 3b). Note that a period-1 orbit as in Fig. 3a is qualitatively different from a period-2 orbit as in Fig. 3b. To visualize it, consider a rubber-band of the shape like in Fig. 3a. Can you transform it into a shape like in Fig. 3b by quantitatively pulling or pushing any part of it? You cannot. In order to transform one to the other, you have to *fold* it once. In that sense these two figures are not "topologically equivalent."

In the same way, there may be trajectories of higher periodicities, each of which is qualitatively different in the above sense.

There can be another type of qualitatively different trajectory. The orbit of moon around earth is periodic, and that of the earth around the sun is also periodic. But what is the trajectory of the moon *around the sun*? It will be a combination of two periodicities. Such orbits in the state space are called "quasiperiodic" orbits (See Fig. 3c).

And if a system's trajectory is bounded but has no periodicity, then it is called *chaos*[1]. A representative example of such an orbit is shown in Fig. 3d.

All these attractors of different periodicities, and the one without any periodicity, represent fundamentally different types of dynamical behaviour.

Therefore, the point of our investigation boils down to the question: What is the mechanism of the transition from one type of trajectory to another?

4. Poincaré section

In attacking this problem, scientists routinely make use of an important technique introduced by the famous mathematician Henri Poincaré about a hundred years back. Imagine that you have placed a plane in the state space such that the trajectory intersects it in every cycle (Fig. 4). Now imagine that you are observing only the points in that plane (called the Poincaré section) where the trajectory pierces it from one side, ignoring what happens elsewhere in the state space. You will see a succession of points — a point *mapping* to another point, that point mapping to another point, so on and so forth.

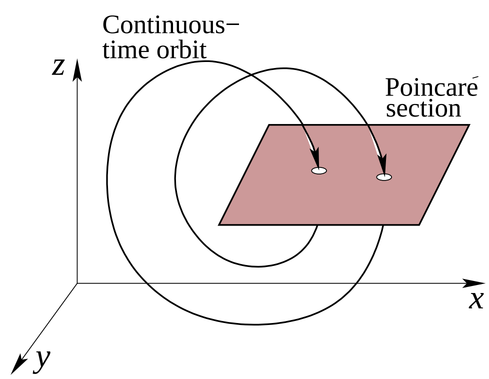


Figure 4: Poincaré section.

Notice that what is seen on the Poincaré section preserves the fundamental character of the original orbit. If the original orbit was period-1, we'll see only one point on the Poincaré section; if it is period-2 we'll see two points; if it is period-n we'll see n points. And if the original orbit does not

[1]The term *chaos* is just a scientific term implying a specific type of orbit, a specific type of dynamical behaviour (bounded, aperiodic trajectory with sensitive dependence on initial condition). It has nothing to do with the common meaning of the word in English language.

From the Breakthrough archives

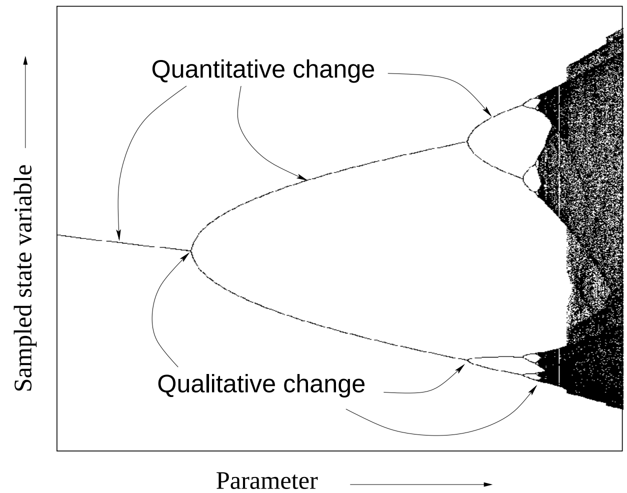


Figure 5: The bifurcation diagram for the system $x_{n+1} = \rho - x_n^2 + 0.3x_n$, $y_{n+1} = x_n$ with the parameter ρ varied over the range 1 to 2.12.

have any periodicity, that is, if it is chaotic, then we'll see an infinity of points on the Poincaré section.

This allows us to pinpoint the event where a fundamental change in the character of an orbit occurs. Suppose we are varying a parameter of a system causing the orbit to change. Take one value of the parameter, let the dynamics settle into a stable orbit, and then plot along the y-axis one coordinate of about 100 points on the Poincaré section. Then take the next value of the parameter and repeat the same procedure. Thus we obtain a plot with the parameter value in the x-axis and the discretely observed value of the state variable in the y-axis. One such plot, called the *bifurcation diagram*, is shown in Fig. 5.

This graph tells a lot of story. For the parameter values where the trajectory is periodic, all the 100 points will fall on the same location, and we'll see just one point. Where the orbit is period-2, 50 points will fall in one position while the other 50 will fall at a different location. We will thus see two points for that parameter value. When the system becomes chaotic, all the 100 points will fall at different locations (because the system then has no periodicity) and we see a smudge of dots. The graph clearly shows the character of the orbit for every parameter value. It would also clearly show the points where a qualitative change in the orbit occurred. In Fig. 5 we have marked the ranges of parameter values where the changes can be said to be quantitative, and the points where qualitative changes occur.

The qualitative changes in a system's dynamical behaviour are called *bifurcations*.

5. Bifurcation

So far so good. We have identified the points where the qualitative changes occur. Now the question is: How do they occur?

It is obvious that so long as a particular orbit is stable, only quantitative changes can occur in response to change of a parameter. Only when it becomes unstable, it can be replaced by a qualitatively different stable orbit.

In any physical system there is a constant interplay of two opposing tendencies — one trying to maintain stability and another trying to destabilize it. What these forces are depends on the particular system under consideration. In an electrical circuit, the exact nature of these forces will not be the same as that in a biological system. But in every system there is a contradiction between two opposing forces, and the stability of a particular dynamical behaviour depends on which tendency becomes dominant. The variation of a parameter changes the balance of forces, leading first to quantitative change, and when the contradiction reaches a nodal point, the earlier orbit loses stability making way for a new trajectory. It is then that we say a *bifurcation* has occurred.

There are mathematical indicators of the balance of forces. In order to visualize it, let us first understand the dynamics on the Poincaré surface, where the nth point maps to the $(n+1)$th point as

$$(x_{n+1}, y_{n+1}) \quad = \quad \text{a function of } (x_n, y_n)$$

In general, the starting point (x_n, y_n) and the point where it lands (x_{n+1}, y_{n+1}) are different. But there will be some special points where $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, which means that if the initial condition is at that point, it will forever remain there. Such points are called *fixed points*.

If the initial condition is somewhere else, the further iterates may converge on to the fixed point; in that case it is stable. If it diverges away from the fixed point, then it is unstable. So long as a fixed point is stable, the orbit is stable, and we can have only quantitative changes. And qualitative change will be associated with the loss of stability of a fixed point.

So let us take a closer look at the fixed point, given by $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, whose stability is under inspection. In the neighbourhood of the fixed point, the function

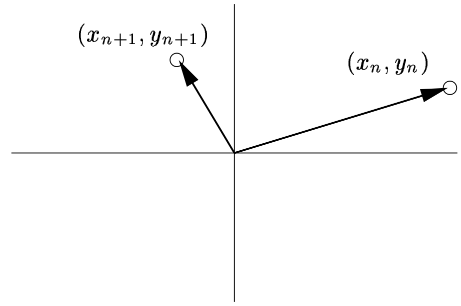


Figure 6: A mapping in the discrete state space.

can be approximated by the linear form

$$\begin{aligned} x_{n+1} &= a\,x_n + b\,y_n, \\ y_{n+1} &= c\,x_n + d\,y_n. \end{aligned}$$

The point (x_n, y_n) represents a vector (imagine a vector joining the origin and this point), so does (x_{n+1}, y_{n+1}) (see Fig. 6). In general, the source vector and the resulting vector lie in different directions. But there are two special directions in a two-dimensional discrete state space such that if (x_n, y_n) happens to be in that direction, the vector (x_{n+1}, y_{n+1}) also lies in the same direction. Any vector along such special directions are called *eigenvectors*. And the factor by which an eigenvector elongates or shortens is called the *eigenvalue.* These can be determined from the equations or from experimental data obtained from practical systems.

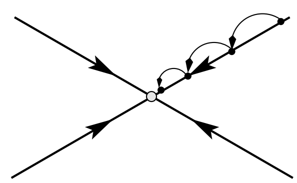


Figure 7: An attracting fixed point: eigenvalues real, $0 < \lambda_1, \lambda_2 < 1$.

Now suppose a system has both the

eigenvalues less than one. If an initial condition is on an eigenvector, in the next iterate it will land closer to the origin (because the vector is multiplied by a number less than one). In subsequent iterates it will move closer and closer (Fig. 7). If an initial condition does not lie on an eigenvector, its coordinate can be decomposed into two components along the eigenvectors. These components will become smaller in every step, and the point will move closer to the fixed point. This implies that the fixed point will be stable.

If both the eigenvalues are greater than one, Fig. 8 shows that the fixed point will be unstable.

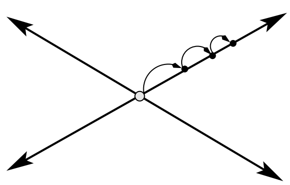


Figure 8: A repeller: eigenvalues real, $\lambda_1, \lambda_2 > 1$.

If one eigenvalue is greater than one and the other less than one, then the system is stable along one eigenvector and unstable along the other. Such a fixed point is called a *saddle.* (Fig. 9).

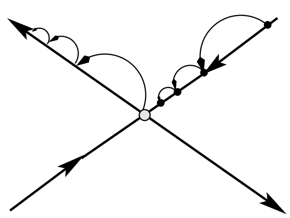


Figure 9: A regular saddle: eigenvalues real, $0 < \lambda_1 < 1$, $\lambda_2 > 1$.

If an eigenvalue is negative, then the vector is multiplied by a negative number. This means that the point flips and lands on the other side of the eigenvector. One such example, the *flip saddle*, is shown in Fig. 10.

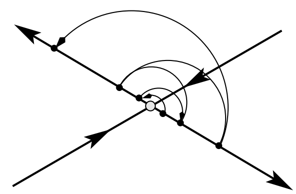


Figure 10: A flip saddle: eigenvalues real, $0 < \lambda_1 < 1$, $\lambda_2 < -1$.

Eigenvalues can also be complex numbers. In that case if the magnitude of the eigenvalue is less than one, then the orbit moves in an inward spiral and converges on the fixed point. If the modulus is greater than one, it travels in an outward spiral and diverges away from the fixed point (Fig. 11).

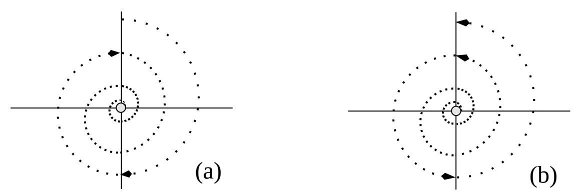


Figure 11: (a) A spiral attractor: eigenvalues complex, $|\lambda_1|, |\lambda_2| < 1$. (b) A spiral repeller: eigenvalues complex, $|\lambda_1|, |\lambda_2| > 1$.

What do all this lead to? What is the condition for stability of a fixed point? Simple. If the magnitudes of all the eigenvalues are less than unity, the fixed point is stable. Else it is unstable.

In how many possible ways can a fixed point lose stability? Exactly three. First, a negative eigenvalue can become less than -1; second, a positive eigenvalue can become greater than $+1$; and third, the modulus of a pair of complex conjugate eigenvalues can become greater than one. These are the three possible ways a qualitative change can occur — three possible bifurcations.

The first one is called period-doubling bifurcation (like Fig. 3a changing to Fig. 3b), examples of which are also seen in Fig. 5. The second one is called saddle-node bifurcation, through which a new orbit, a new dynamical behavour, can come into being or go out of being. The third one is called the Hopf bifurcation, through which a periodic orbit (Fig. 3(a)) can change into a quasiperiodic orbit (Fig. 3(c)). These are the three main mechanisms leading to qualitative change in a system's dynamical behaviour.

Scientists studying the diseases of the heart have found that the stable periodic orbit generally loses stability through period-doubling bifurcation. People have observed the dynamics on the Poincaré section, obtained the eigenvalues, and have confirmed that one eigenvalue does become -1 at the point where instability sets in, leading to a qualitative change in the behavior. This knowledge is now helping scientists devise diagnostic techniques and preventive measures.

In very large electric power systems sometimes an instability sets in leading to a voltage collapse. This happens not only in weakly protected power systems of our country, but also in relatively stronger systems of advanced countries. Scientists probing this sudden qualitative change in the system's behaviour have found out that a type of Hopf bifurcation is the root cause of the phenomenon.

Engineers designing jet engines were baffled by a peculiar phenomenon of sudden stalling of the engine under certain operating conditions. As a parameter, say the air intake or fuel injection, is continuously varied, a quantitative change in the operating condition takes place. But sometimes, at a nodal point, the operating condition suddenly undergoes a qualitative change. One can easily imagine the danger of such a thing happening mid-air, and so it has received a considerable amount of research attention. It has been found out that a bifurcation phenomenon is the culprit in this case also.

Ecologists studying why populations of certain species fluctuate in cyclic rhythms have mathematically modeled the complex struggle of the species with its environment, and found that this is precisely what is to be expected, as per the bifurcation theory.

Even mundane events — like the flow of drops of water from a leaky faucet — exhibit bifurcation phenomena. Suppose, you have opened a tap a little bit, such that drops of water fall down: tick tick tick $\cdots$ in equal intervals. In scientific perlance, it will be called a period-1 orbit. Now increase the flow a little bit, and it becomes period-2. If you have means of controlling the flow precisely, you will see the same kind of bifurcation structure as seen in Fig. 5, finally leading to chaos — erratic and aperiodic fall of the drops. Again, scientists have computed the eigenvalues from experimental observations and have found that here also the qualitative changes in the orbit follow the mathematical conditions outlined above.

Internal and external

The eigenvalues of a system can indicate the balance of forces that are internal to the system. Naturally, they can identify only those bifurcations that are caused by internal contradictions. Sometimes, some extraneous causes can abruptly destabilize a system, causing a bifurcation.

Take a simple example from a physical system (Fig. 12). Suppose there is a mass attached to a wall by means of a spring. If you apply a force to the mass periodically, it will oscillate in the horizontal direction. You can easily write down the equations and can show that the system will be sta-

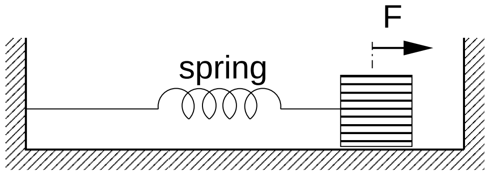


Figure 12: The "impact oscillator."

ble. But now suppose there is another wall at some distance, so that if the amplitude of oscillation crosses a critical value the mass hits the wall. This extraneous cause destabilizes the periodic orbit and causes a bifurcation.

Similar events occur in all systems where the normal balance of forces undergo some abrupt change when some condition is satisfied. Examples include systems involving some switching action, relays, valve opening and closing etc., and systems whose environments exert some influence under specific conditions.

For such systems it has been shown that when the discrete-time representation is obtained (by Poincaré section), one obtains different functional forms of the map in different regions of the state space. There are *borderlines* that divide these regions. Specific types of bifurcations can occur when a fixed point collides with one such borderline, causing an abrupt change in the eigenvalues. Such bifurcations are called "border collision bifurcations." Mathematical analysis of such bifurcations is helping scientists understand the cause of many sudden changes that happen in natural and engineering systems, including the excitation and inhibition of the neurons of the brain, abnormal conditions of the human heart and traffic congestion in highways, even internet traffic in computer networks.

Conclusion

From the above discussion we see that the philosophical conclusion regarding the process of change in material phenomena, reached a century and half ago, now stands on solid scientific ground. Rigorous experimental tests have been carried out to investigate the transition from quantitative change to qualitative change in natural processes, and mathematical theories to explain these have been developed. Armed with this knowldege, scientists are now in a position to exercise a greater control over the process of change in natural as well as engineering systems. □

Further Reading:

1. A popular book on the subject is "Chaos: The Making of a New Science" by James Gleick, published by Viking Penguin, 1987.
2. Some good technical books are (1) "Nonlinear Dynamics and Chaos" by S. H. Strogatz, published by Addison-Wesley, 1994. (2) "Chaos in Dynamical Systems" by E. Ott, published by Cambridge, 1993. (3) "Chaos: An Introduction to Dynamical Systems" by K. T. Alligood, T. D. Sauer and J. A. Yorke, published by Springer-Verlag, 1996.
3. Some good review papers are R. S. Shaw, J. P. Crutchfield, J. Doyne Farmer, N. H. Packard, "Chaos", *Scientific American*, Vol.255, 1986, p46 (the dripping faucet experiment can be found here); Robert M. May, "Simple Mathematical Models with Very Complicated Dynamics," *Nature*, Vol.261, 1976, p.459 (Reference to population biology can be found here). For the nonlinear dynamics of power systems, see I. Dobson and H. D. Chiang, "Towards a theory of voltage collapse in electric power systems," *Systems & Control Letters*, Vol.13, 1989, p.253. Analysis of border collision bifurcations can be found in S. Banerjee and C. Grebogi, "Border collision bifurcations in two-dimensional piecewise smooth maps," *Physical Review E*, Vol. 59, 1999, p.4052.